Boundedness

The concept of *boundedness* is fairly simple to understand both graphically and algebraically. We will move directly to the algebraic definition after motivating the concept with some typical graphs (Figure 1.22).

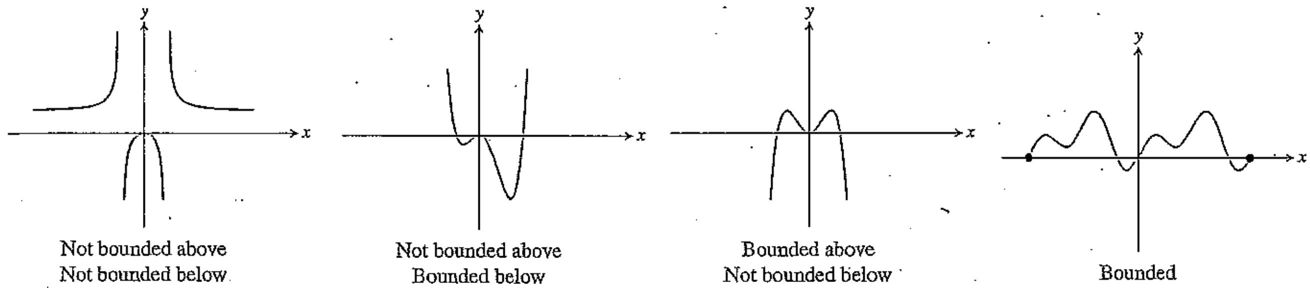


FIGURE 1.22 Some examples of graphs bounded and not bounded above and below.

DEFINITION Lower Bound, Upper Bound, and Bounded

A function f is **bounded below** if there is some number b that is less than or equal to every number in the range of f. Any such number b is called a **lower bound** of f.

A function f is **bounded above** if there is some number B that is greater than or equal to every number in the range of f. Any such number B is called an **upper bound** of f.

A function f is **bounded** if it is bounded both above and below.

We can extend the above definition to the idea of **bounded on an interval** by restricting the domain of consideration in each part of the definition to the interval we wish to consider. For example, the function $f(x) = 1/x$ is bounded above on the interval $(-\infty, 0)$ and bounded below on the interval $(0, \infty)$.

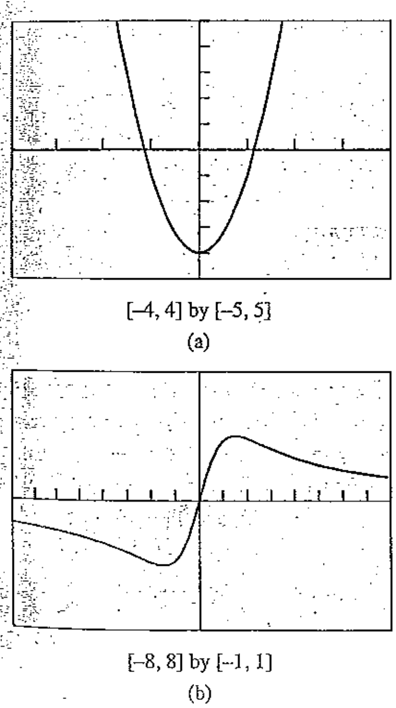


FIGURE 1.23 The graphs for Example 7. Which are bounded where?

EXAMPLE 7 Checking Boundedness

Identify each of these functions as bounded below, bounded above, or bounded.

(a) $w(x) = 3x^2 - 4$ **(b)** $p(x) = \dfrac{x}{1 + x^2}$

SOLUTION

Solve Graphically

The two graphs are shown in Figure 1.23. It appears that w is bounded below, and p is bounded.

Confirm Graphically

We can confirm that w is bounded below by finding a lower bound, as follows:

$$\begin{aligned} x^2 &\geq 0 && x^2 \text{ is nonnegative} \\ 3x^2 &\geq 0 && \text{Multiply by 3.} \\ 3x^2 - 4 &\geq 0 - 4 && \text{Subtract 4.} \\ 3x^2 - 4 &\geq -4 \end{aligned}$$

Thus, -4 is a lower bound for $w(x) = 3x^2 - 4$.

We leave the verification that p is bounded as an exercise (Exercise 77).

Now try Exercise 37.

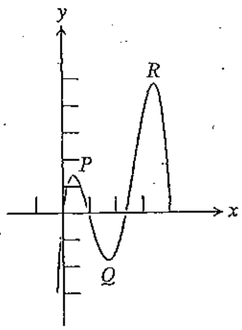


FIGURE 1.24 The graph suggests that f has a local maximum at P, a local minimum at Q, and a local maximum at R.

Local and Absolute Extrema

Many graphs are characterized by peaks and valleys where they change from increasing to decreasing and vice versa. The extreme values of the function (or *local extrema*) can be characterized as either *local maxima* or *local minima*. The distinction can be easily seen graphically. Figure 1.24 shows a graph with three local extrema: local maxima at points P and R and a local minimum at Q.

This is another function concept that is easier to see graphically than to describe algebraically. Notice that a local maximum does not have to be *the* maximum value of a function; it only needs to be the maximum value of the function on *some* tiny interval.

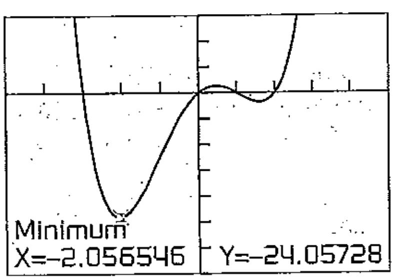
We have already mentioned that the best method for analyzing increasing and decreasing behavior involves calculus. Not surprisingly, the same is true for local extrema. We will generally be satisfied in this course with approximating local extrema using a graphing calculator, although sometimes an algebraic confirmation will be possible when we learn more about specific functions.

DEFINITION Local and Absolute Extrema

A **local maximum** of a function f is a value $f(c)$ that is greater than or equal to all range values of f on some open interval containing c. If $f(c)$ is greater than or equal to all range values of f, then $f(c)$ is the **maximum** (or **absolute maximum**) value of f.

A **local minimum** of a function f is a value $f(c)$ that is less than or equal to all range values of f on some open interval containing c. If $f(c)$ is less than or equal to all range values of f, then $f(c)$ is the **minimum** (or **absolute minimum**) value of f.

Local extrema are also called **relative extrema**.



[–5, 5] by [–35, 15]

FIGURE 1.25 A graph of $y = x^4 - 7x^2 + 6x$. (Example 8)

EXAMPLE 8 Identifying Local Extrema

Decide whether $f(x) = x^4 - 7x^2 + 6x$ has any local maxima or local minima. If so, find each local maximum value or minimum value and the value of x at which each occurs.

SOLUTION The graph of $y = x^4 - 7x^2 + 6x$ (Figure 1.25) suggests that there are two local minimum values and one local maximum value. We use the graphing calculator to approximate local minima as -24.06 (which occurs at $x \approx -2.06$) and -1.77 (which occurs at $x \approx 1.60$). Similarly, we identify the (approximate) local maximum as 1.32 (which occurs at $x \approx 0.46$).

Now try Exercise 41.

USING A GRAPHER TO FIND LOCAL EXTREMA

Most modern graphers have built in "maximum" and "minimum" finders that identify local extrema by looking for sign changes in Δy. It is not easy to find local extrema by zooming in on them, as the graphs tend to flatten out and hide the very behavior you are looking at. If you use this method, keep narrowing the vertical window to maintain some curve in the graph.

Symmetry

In the graphical sense, the word "symmetry" in mathematics carries essentially the same meaning as it does in art: The picture (in this case, the graph) "looks the same" when viewed in more than one way. The interesting thing about mathematical symmetry is that it can be characterized numerically and algebraically as well. We will be

SECTION 1.2 EXERCISES

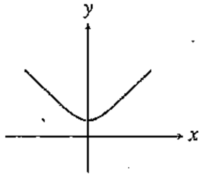
In Exercises 1–4, determine whether the formula determines y as a function of x. If not, explain why not.

1. $y = \sqrt{x - 4}$ **2.** $y = x^2 \pm 3$

3. $x = 2y^2$ **4.** $x = 12 - y$

In Exercises 5–8, use the vertical line test to determine whether the curve is the graph of a function.

5.



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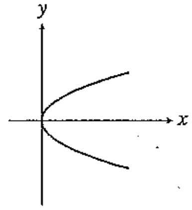

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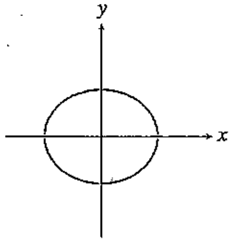

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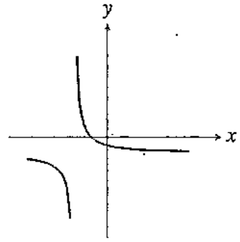


In Exercises 9–16, find the domain of the function algebraically and support your answer graphically.

9. $f(x) = x^2 + 4$ **10.** $h(x) = \dfrac{5}{x - 3}$

11. $f(x) = \dfrac{3x - 1}{(x + 3)(x - 1)}$ **12.** $f(x) = \dfrac{1}{x} + \dfrac{5}{x - 3}$

13. $g(x) = \dfrac{x}{x^2 - 5x}$ **14.** $h(x) = \dfrac{\sqrt{4 - x^2}}{x - 3}$

15. $h(x) = \dfrac{\sqrt{4 - x}}{(x + 1)(x^2 + 1)}$ **16.** $f(x) = \sqrt{x^4 - 16x^2}$

In Exercises 17–20, find the range of the function.

17. $f(x) = 10 - x^2$ **18.** $g(x) = 5 + \sqrt{4 - x}$

19. $f(x) = \dfrac{x^2}{1 - x^2}$ **20.** $g(x) = \dfrac{3 + x^2}{4 - x^2}$

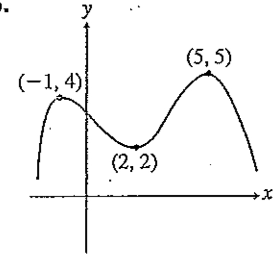
In Exercises 21–24, graph the function and tell whether or not it has a point of discontinuity at $x = 0$. If there is a discontinuity, tell whether it is removable or nonremovable.

21. $g(x) = \dfrac{3}{x}$ **22.** $h(x) = \dfrac{x^3 + x}{x}$

23. $f(x) = \dfrac{|x|}{x}$ **24.** $g(x) = \dfrac{x}{x - 2}$

In Exercises 25–28, state whether each labeled point identifies a local minimum, a local maximum, or neither. Identify intervals on which the function is decreasing and increasing.

25.



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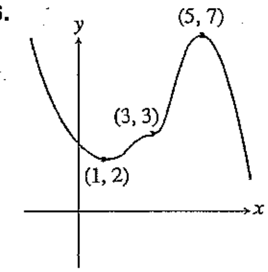

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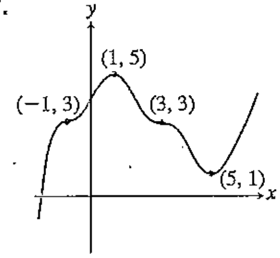

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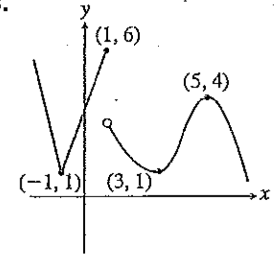


In Exercises 29–34, graph the function and identify intervals on which the function is increasing, decreasing, or constant.

29 $f(x) = |x + 2| - 1$

30. $f(x) = |x + 1| + |x - 1| - 3$

31. $g(x) = |x + 2| + |x - 1| - 2$

32. $h(x) = 0.5(x + 2)^2 - 1$

33. $g(x) = 3 - (x - 1)^2$

34. $f(x) = x^3 - x^2 - 2x$

In Exercises 35–40, determine whether the function is bounded above, bounded below, or bounded on its domain.

35. $y = 32$ **36.** $y = 2 - x^2$

37. $y = 2^x$ **38.** $y = 2^{-x}$

39. $y = \sqrt{1 - x^2}$ **40.** $y = x - x^3$

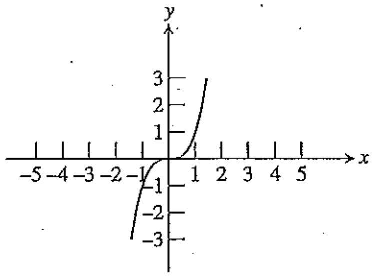
In Exercises 41–46, use a grapher to find all local maxima and minima and the values of x where they occur. Give values rounded to two decimal places.

41. $f(x) = 4 - x + x^2$ **42.** $g(x) = x^3 - 4x + 1$

43. $h(x) = -x^3 + 2x - 3$ **44.** $f(x) = (x + 3)(x - 1)^2$

45. $h(x) = x^2\sqrt{x + 4}$ **46.** $g(x) = x|2x + 5|$

The Cubing Function

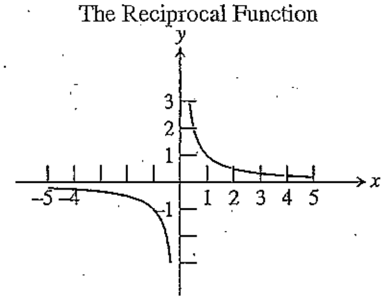


$f(x) = x^3$

Interesting fact: The origin is called a "point of inflection" for this curve because the graph changes curvature at that point.

FIGURE 1.38

The Reciprocal Function

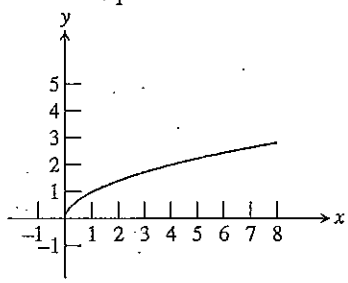


$f(x) = \frac{1}{x}$

Interesting fact: This curve, called a hyperbola, also has a reflection property that is useful in satellite dishes.

FIGURE 1.39

The Square Root Function

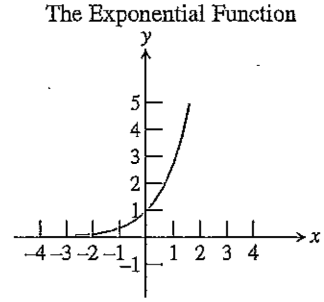


$f(x) = \sqrt{x}$

Interesting fact: Put any positive number into your calculator. Take the square root. Then take the square root again. Then take the square root again, and so on. Eventually you will always get 1.

FIGURE 1.40

The Exponential Function

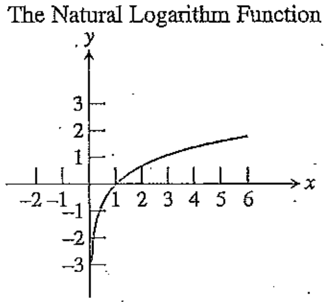


$f(x) = e^x$

Interesting fact: The number e is an irrational number (like π) that shows up in a variety of applications. The symbols e and π were both brought into popular use by the great Swiss mathematician Leonhard Euler (1707–1783).

FIGURE 1.41

The Natural Logarithm Function

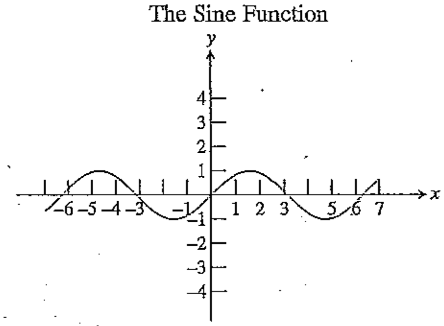


$f(x) = \ln x$

Interesting fact: This function increases very slowly. If the x-axis and y-axis were both scaled with unit lengths of one inch, you would have to travel more than two and a half miles along the curve just to get a foot above the x-axis.

FIGURE 1.42

The Sine Function



$f(x) = \sin x$

Interesting fact: This function and the sinus cavities in your head derive their names from a common root: the Latin word for "bay." This is due to a 12th-century mistake made by Robert of Chester, who translated a word incorrectly from an Arabic manuscript.

FIGURE 1.43