Equation Solving and Modeling

Chapter 3 Section 5

Quick Review

Prove that each function in the given pair is the inverse of the other.

1. $f(x) = e^{3x}$ and $g(x) = \ln\left(x^{1/3}\right)$

2. $f(x) = \log x^2$ and $g(x) = 10^{x/2}$

Write the number in scientific notation.

3. 123,400,000

Write the number in decimal form.

4. 5.67×10^8

5. 8.91×10^{-4}

Quick Review Solutions

1. $f(x)=e^{3x}$ and $g(x)=\ln\left(x^{1/3}\right)$

$$f(g(x))=e^{3\ln\left(x^{1/3}\right)}=e^{\ln(x)}=x$$

2. $f(x)=\log x^{2}$ and $g(x)=10^{x/2}$

$$f(g(x))=\log\left(10^{x/2}\right)^{2}=\log 10^{x}=x$$

3. $123,400,000$ $\quad 1.234\times 10^{8}$

4. 5.67×10^{8} $\quad 567,000,000$

5. 8.91×10^{-4} $\quad 0.000891$

What you'll learn about

Solving Exponential Equations
Solving Logarithmic Equations
Orders of Magnitude and Logarithmic Models
Newton's Law of Cooling (Enrichment Applications)
Logarithmic Re-expression (Enrichment Regression)

... and why

The Richter scale, pH, and Newton's Law of Cooling, are among the most important uses of logarithmic and exponential functions.

Today's Objectives

CO: Construct the equation necessary to solve exponential function problems involving radioactive decay.

Success Criteria

- Solve equations using the one-to-one properties of exponential and logarithmic functions
- Define radioactive decay
- Compare exponential and logarithmic functions to orders of magnitude

LO: Read words problems concerning radioactive decay and decipher the real-world meaning using CUS. Write solutions to word problems terms of the situation using TAG'M.

Vocabulary: one-to-one, order of magnitude

One-to-One Properties

For any exponential function $f(x) = b^x$,

if $b^u = b^v$, then $u = v$.

For any logarithmic function $f(x) = \log_b x$,

if $\log_b u = \log_b v$, then $u = v$.

Example: Using the One to One property

$$40(1/2)^{x/2} = 5$$

1. Find the x value that satisfies the exponential equation.

$$\left(\frac{1}{2}\right)^{x/2} = \frac{1}{8}$$

2. Simplify as much as possible using algebraic properties

$$\left(\frac{1}{2}\right)^{x/2} = \frac{1}{2^3}$$

3. Look for a common base and rewrite all expression according to the common base.

$$\left(\frac{1}{2}\right)^{x/2} = \left(\frac{1}{2}\right)^3$$

4. Use the properties of exponents to get all exponents into the same position.
5. Use the one to one property to equate the exponents

$$x/2 = 3$$

6. Solve for x.

$$x = 6$$

Example: Solving a Logarithmic Equation

$$8 - \frac{5}{3}\log x^3 = 3$$

$$-\frac{5}{3}\log x^3 = -5$$

$$\log x^3 = 3$$

$$10^{\log x^3} = 10^3$$

$$x^3 = 10^3$$

$$x = 10$$

1. Begin to solve the logarithmic equation by simplifying the expression by appropriate algebraic steps.
2. Apply the inverse function to both sides of the equation by making both sides into exponents for the base 10.
3. Exponential functions undo logarithmic function.
4. Solve algebraically

AM: Solve Logarithmic Equations

1. Find all real solutions of the following equation: $\log_3(x+3)+\log_3(x+9)=3$

[A] $x=0$ [B] $x=3,\ 9$ [C] $x=9$ [D] none of these

$$\log_3(x+3)+\log_3(x+9)=3$$

1. Identify any log addition, subtractions or exponents.
2. Apply the log product rule since addition of same base logs occurs.

$$\log_3\left[(x+3)(x+9)\right]=3$$

3. Undo log with appropriate inverse exponential function.

$$3^{\log_3\left[(x+3)(x+9)\right]}=3^3$$

$$\left[(x+3)(x+9)\right]=27$$

4. Simplify the expression
5. Solve for x using any algebraic means necessary

$$x^2+12x+27=27$$

$$x(x+12)=0, x=0,-12$$

6. Check for extraneous solutions by verifying solution in the original equation

$$\log_3(0+3)+\log_3(0+9)=3$$

$$\log_3(-12+3)+\log_3(-12+9)\ \neq 3$$

AM: Solve Logarithmic Equations

1. Find all real solutions of the following equation: $\log_3(x+3)+\log_3(x+9)=3$

[A] $x=0$ [B] $x=3,\ 9$ [C] $x=9$ [D] none of these

LO: First, I can use the product rule to condense the added terms into the equivalent multiplied form. This will give me the equation ________________________.

I can now change the equation to exponential form, using a base of 3. This will give me the equation ____________.

Lastly, I solve for x, which gives me an answer of _____.

AM: Solve Logarithmic Equations

Solve:

2. $\log_3 x = 5$ [A] 125 [B] 81 [C] 243 [D] none of these

LO: I can change the equation to exponential form, using a base of 3. This will give me the equation ____________.

Lastly, I solve for x, which gives me an answer of _____.

AM: Solve Logarithmic Equations

3. $\log_{256} x = -\frac{5}{4}$ [A] $\sqrt[4]{1024}$ [B] $\sqrt[5]{256}$ [C] $\frac{1}{1024}$ [D] 1024

LO: I can change the equation to exponential form, using a base of 256. This will give me the equation ____________.

Lastly, I solve for x, which gives me an answer of _____.

AM: Solve Logarithmic Equations

4. Which is the value of x? [A] $\frac{1}{27}$ [B] 6 [C] 27 [D] 3

$\log_x 9 = -\frac{2}{3}$

LO: I can change the equation to exponential form, using a base of x. This will give me the equation ________.

Next, I use exponent and radical rules to solve for x, which equals _____.

AM: Solve Exponential Equations **using One to One Property**

2. Solve: $\frac{1}{9} = 27^{4x-3}$ [A] $\frac{11}{12}$ [B] $\frac{1}{12}$ [C] $\frac{1}{4}$ [D] $\frac{7}{12}$

$$\frac{1}{9} = 27^{4x-3}$$

1. Find the x value that satisfies the exponential equation.

$$\frac{1}{3^2} = \left(3^3\right)^{4x-3}$$

2. Look for a common base and rewrite all expression according to the common base. **This will not always be possible, but it will work for this problem.**

$$(3)^{-2} = (3)^{12x-9}$$

3. Use the properties of exponents to get all exponents into the same position.

$$-2 = 12x - 9$$

4. Use the one to one property to equate the exponents

$$x = \frac{7}{12}$$

5. Solve for x.

AM: Solve Exponential Equations **using Logarithms to undo exponents**

$$\frac{1}{9} = 27^{4x-3}$$

$$\ln\left(\frac{1}{9}\right) = \ln\left(27^{4x-3}\right)$$

$$\ln\left(\frac{1}{9}\right) = (4x-3)\ln(27)$$

$$4x-3 = \frac{\ln\left(\frac{1}{9}\right)}{\ln(27)}$$

$$4x = \frac{\ln\left(\frac{1}{9}\right)}{\ln(27)} + 3$$

$$x = \frac{\frac{\ln\left(\frac{1}{9}\right)}{\ln(27)} + 3}{4} = .58\bar{3} = \frac{7}{12}$$

1. **Alternate method apply ln().** Find the x value that satisfies the exponential equation.
2. Use logarithms and there properties to get variable expressions out of the exponent position. **This method will always be possible, but a calculator is usually but not always needed.**
3. Treat log expressions that have numbers as inputs as the real numbers they are. Do not change to decimal form you will use precision by rounding in the middle of a problem
4. Use appropriate properties of algebra to isolate x.
5. Solve for x, by evaluating the expression with your calculator. Use appropriate grouping symbols to ensure the order of operation is correct.

AM: Solve Exponential Equations **using Logarithms to undo exponents**

9

$$\frac{1}{9} = 27^{4x-3}$$

$$\log\left(\frac{1}{9}\right) = \log\left(27^{4x-3}\right)$$

$$\log\left(\frac{1}{9}\right) = (4x-3)\log(27)$$

1. **Alternate method apply log().** Find the x value that satisfies the exponential equation.
2. Use logarithms and there properties to get variable expressions out of the exponent position. **This method will always be possible, but a calculator is usually but not always needed.**

$$4x - 3 = \frac{\log\left(\frac{1}{9}\right)}{\log(27)}$$

3. Treat log expressions that have numbers as inputs as the real numbers they are. Do not change to decimal form you will use precision by rounding in the middle of a problem

$$4x = \frac{\log\left(\frac{1}{9}\right)}{\log(27)} + 3$$

4. Use appropriate properties of algebra to isolate x.

$$x = \frac{\frac{\log\left(\frac{1}{9}\right)}{\log(27)} + 3}{4} = .58\bar{3} = \frac{7}{12}$$

5. Solve for x, by evaluating the expression with your calculator. Use appropriate grouping symbols to ensure the order of operation is correct.

Turn and Talk:
Ideas about Logarthims

$$\frac{1}{9} = 27^{4x-3}$$

- In the previous three slides we used three different processes to solve the same equation.
- We choose to use *ln()* and *log()* more frequently then other logarithm functions because *ln()* is the inverse of the natural base *e* and the common log is the obvious choice when working with powers of 10, hence the calculator buttons for these exponents and logs.
- Working with powers of 10 is **common** because we have a base 10 system, {0,1,2,3,4,5,6,7,8,9} and we form our every day numbers as powers of 10. Scientific notation states numbers as powers of ten.
- Consider the first method. We used the one to one property and the fact that both 9 and 27 are powers of three, that is 3^2 and 3^3 respectively.
- Despite these reasons we could have chosen any logarithmic function we like to solve the problem using logs.
- **Report to me:** The best logarithm function to apply in this instance was not ln() or log(). What logarithmic function would have made the most sense? Why?

Last Method: Solve the problem using the best logarithm function

2. Solve: $\frac{1}{9} = 27^{4x-3}$ [A] $\frac{11}{12}$ [B] $\frac{1}{12}$ [C] $\frac{1}{4}$ [D] $\frac{7}{12}$

AM: Solve Exponential Equations

4. Solve for x to the nearest hundredth: $6^x = 7$

LO: First, I take the ___________of both sides to get the equation _____________________. Using the exponent rule, I can rewrite the equation using the equivalent coefficients into _________________________________.

Lastly, I solve for x, which gives me an answer of ______________.

AM: Solve Exponential Equations

1. Find the solution: $2.43^x = 32$ [A] 3.90 [B] 0.39 [C] 0.26 [D] 1.51

LO: First, I take the _________ of both sides to get the equation _____________________. Using the exponent rule, I can rewrite the equation using the equivalent coefficients into

_________________________________.

Lastly, I solve for x, which gives me an answer of ______________.

AM: Solve Exponential Equations

2. Solve: $\frac{1}{8} = 4^{5x-5}$ [A] $\frac{1}{5}$ [B] $\frac{13}{10}$ [C] $\frac{2}{5}$ [D] $\frac{7}{10}$

LO: I can rewrite each expression with a base of _______.
This will give me the equation _________________.
I can now use the equivalent exponent rules to set up a linear equation, which is _____________.
Lastly, I solve for x, which gives me an answer of ______.

3. Find the solution: $5^x = 8^{x+7}$

[A] -15.4851 [B] 3.9459 [C] -30.9702 [D] -18.6667

$$5^x = 8^{x+7}$$

$$\ln\left(5^x\right) = \ln\left(8^{x+7}\right)$$

$$x\ln(5) = (x+7)\ln(8)$$

$$x\ln(5) = x\ln(8) + 7\ln(8)$$

$$x\ln(5) - x\ln(8) = 7\ln(8)$$

$$x(\ln(5) - \ln(8)) = 7\ln(8)$$

$$x = \frac{7\ln(8)}{\ln(5) - \ln(8)}$$

$$x = -30.97016670$$

$$x \approx -30.9702$$

LO: First, I take the common log of both sides to get the equation ________________.

Using the exponent rule, I can rewrite the equation using the equivalent coefficients into ________ _________________________ _________________________.

Lastly, I solve for x, which gives me an answer of_______________ _________________________ _________________________ .

AM: **Solve Exponential Equations**

1. Solve the equation $3^{x-6} = 5^{2x+5}$ and select the answer expressed in terms of natural logarithms.

[A] $x = \dfrac{5\ln 5 + 6\ln 3}{\ln 3 - 2\ln 5}$

[B] $x = \ln\left[\dfrac{5^5}{3^{-6}} - \dfrac{3}{5^2}\right]$

[C] $x = \dfrac{5\ln 3 - 6\ln 5}{\ln 5 - 2\ln 3}$

[D] none of these

LO: First, I take the ______________of both sides to get the equation _____________________. Using the exponent rule, I can rewrite the equation using the equivalent coefficients into _________________________________. Lastly, I solve for x, which gives me an answer of _______________.

AM: Solve Exponential Equations

4. Solve the equation $2^{x-2} = 3^{2x+3}$ and express the answer in terms of natural logarithms.

LO: First, I take the _____________of both sides to get the equation ____________________. Using the exponent rule, I can rewrite the equation using the equivalent coefficients into _______________________________.

Lastly, I solve for x, which gives me an answer of ______________.

AM: Solve Exponential Equations

3. Solve correct to four decimal places: $6^x = 7^{x+2}$

[A] -14.0000 [B] -12.6234 [C] 1.0412 [D] none of these

LO: First, I take the ____________of both sides to get the equation ____________________. Using the exponent rule, I can rewrite the equation using the equivalent coefficients into

______________________________.

Lastly, I solve for x, which gives me an answer of _____________.

WP: Radioactive Decay

- Exponential functions can also model phenomena that produce a decrease over time, such as happens with radioactive decay.
- The **half-life of a radioactive substance is the amount of time** it takes for half of the substance to change from its original radioactive state to a nonradioactive state by emitting energy in the form of radiation.
- Suppose that we begin with 200 units of radioactive material that decreases by 50% every ten years.
- The rate of decay is .5 and the half life is 10.
- The table which represents this information is populated by counting by half-lives and multiplying the previous output by .5

Half Life Example		
Time in Years	Time in Half Lifes	Amount of Radioactive Material
0	0	Initial Amount = 200
10	1	$200(.5)^1 = 100$
20	2	$200(.5)^2 = 50$
30	3	$200(.5)^3 = 25$
40	4	$200(.5)^4 = 12.5$
50	5	$200(.5)^5 = 6.25$
60	6	$200(.5)^6 = 3.125$
65	6.5	$200(.5)^{6.5} =$
73	7.3	$200(.5)^{7.3} =$
t	?	$200(.5)^{?} =$
Exponential Decay Function=		

WP: Radioactive Decay

1. The half-life of a radioactive element is 133 days, but your sample will not be useful to you after 80% of the radioactive nuclei originally present have disintegrated. About how many days can you use the sample?

[A] 309 [B] 319 [C] 304 [D] 324

$$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{halflife}}$$

$$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{133}}$$

$$.20A_0 = A_0\left(\frac{1}{2}\right)^{\frac{t}{133}}$$

$$.2 = \left(\frac{1}{2}\right)^{\frac{t}{133}}$$

$$t =$$

LO: What are the inputs and outputs of this function?

The inputs for the functions are times measured in __________.
The outputs of this radioactive decay function are the **amounts ____________ of radioactive materials remaining** after the specified time.

WP: Radioactive Decay

3. The half-life of carbon-14 is 5700 years. Find the age of a sample at which 22% of the radioactive nuclei originally present have decayed.

[A] 2043 years [B] 1043 years [C] 1593 years [D] 2143 years

$$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{halflife}}$$

$$A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{5700}}$$

$$.78A_0 = A_0\left(\frac{1}{2}\right)^{\frac{t}{5700}}$$

$$.78 = \left(\frac{1}{2}\right)^{\frac{t}{5700}}$$

$$t =$$

LO: What are the inputs and outputs of this function?

The inputs for the functions are times measured in __________. The outputs of this radioactive decay function are the **amounts ____________ of radioactive materials remaining** after the specified time.

WP: Radioactive Decay

2. A radioactive substance decays so that the amount A present at time t (years) is $A = A_0 e^{-1.8t}$. Find the half-life (time for half to decay) of this substance. $(\ln 2 \approx 0.69315)$

[A] about 0.770 yr [B] about 0.385 yr [C] about 3.465 yr [D] none of these

1. Now that you have the half-life re-model the exponential decay situation using (1/2) as the base?
2. Compare the models graphically with an initial amount of 100?
3. Will the new model hold for any initial amount ?
4. Why is it possible to model half life with base *e*?

WP: Radioactive Decay

4. A certain radioactive material decays according to the law $A = A_0 e^{-0.0244t}$, where A_0 is the initial amount present and A is the amount present in t years. What is the half-life of this material? Round the answer to two decimal places. (Hint: The half-life is the time required for A to be reduced to $\frac{A_0}{2}$.)

Orders of Magnitude

The common logarithm of a positive quantity is its **order of magnitude**.

Orders of magnitude can be used to compare any like quantities:

- A kilometer is 3 orders of magnitude longer than a meter.
- A dollar is 2 orders of magnitude greater than a penny.
- New York City with 8 million people is 6 orders of magnitude bigger than Earmuff Junction with a population of 8.

Richter Scale

The Richter scale magnitude R of an earthquake is $R = \log\frac{a}{T} + B$, where a is the amplitude in micrometers (μm) of the vertical ground motion at the receiving station, T is the period of the associated seismic wave in seconds, and B accounts for the weakening of the seismic wave with increasing distance from the epicenter of the earthquake.

pH

In chemistry, the acidity of a water-based solution is measured by the concentration of hydrogen ions in the solution (in moles per liter). The hydrogen-ion concentration is written $[H^+]$. The measure of acidity used is **pH**, the opposite of the common log of the hydrogen-ion concentration:

$$pH = -\log [H^+]$$

More acidic solutions have higher hydrogen-ion concentrations and lower pH values.

Newton's Law of Cooling

An object that has been heated will cool to the temperature of the medium in which it is placed.

The temperature T of the object at time t can be modeled by $T(t) = T_m + (T_0 - T_m)e^{-kt}$ for an appropriate value of k, where $T_m =$ the temperature of the surrounding medium, $T_0 =$ the temperature of the object.

This model assumes that the surrounding medium maintains a constant temperature.

Example **Newton's Law of Cooling**

A hard-boiled egg at temperature 100°C is placed in 15°C water to cool. Five minutes later the temperature of the egg is 55°C.

When will the egg be 25°C?

A hard-boiled egg at temperature 100°C is placed in 15°C water to cool. Five minutes later the temperature of the egg is 55°C.

When will the egg be 25°C?

Given $T_0 = 100$, $T_m = 15$, and $T(5) = 55$.

$$T(t) = T_m + (T_0 - T_m)e^{-kt}$$

$$55 = 15 + 85e^{-5k}$$

$$40 = 85e^{-5k}$$

$$\left(\frac{40}{85}\right) = e^{-5k}$$

$$\ln\left(\frac{40}{85}\right) = -5k$$

$$k = 0.1507\ldots$$

A hard-boiled egg at temperature 100°C is placed in 15°C water to cool. Five minutes later the temperature of the egg is 55°C.

When will the egg be 25°C?

Now find t when $T(t) = 25$.

$$25 = 15 + 85e^{-0.1507\,t}$$

$$10 = 85e^{-0.1507\,t}$$

$$\ln\left(\frac{10}{85}\right) = -0.1507t$$

$$t = 14.2\,\text{min}.$$